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Kostant's Formula for Kac–Moody Lie Algebras

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This paper is a generalization of H. Garland and J. Lepowsky's paper of 1976. Let $\mathbf{g}(A)$ be the Kac–Moody Lie algebra defined by a symmetrizable generalized Cartan matrix A, $M(\lambda)$ the irreducible highest weight module of $\mathbf{g}(A)$ with a dominant integral highest weight λ. Kostant's homology and cohomology formulas for the Lie algebra $\mathbf{g}(A)$ and the module $M(\lambda)$ are proved without assuming that the reductive part of the parabolic subalgebra of $\mathbf{g}(A)$ is of finite type. A resolution of $M(\lambda)$ is constructed. For any j, the jth term of the resolution has a filtration such that all the factors of the filtration are generalized Verma modules of the form $V^{m(w(\lambda+\rho)-\rho)}$, where w ranges over a certain subset of the Weyl group of $\mathbf{g}(A)$.

Contents.

INTRODUCTION

From Kostant's homology or cohomology formulas, Weyl's character formula can be obtained easily. In 1967, F. Aribaud reversed everything. He started with Weyl's character formula, from which he obtained Weyl-Kostant's formula, which is a generalization of Weyl's character formula. By using this formula and Casimir operators he gave a new proof of Kostant's cohomology formula (see [3]). His paper cast new light on the connections among these formulas.

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In 1976, H. Garland and J. Lepowsky proved Kostant's homology formula for Kac–Moody Lie algebras, under the conditions that A is symmetrizable and the reductive part of the parabolic subalgebra is of finite type. (These are called F-parabolic subalgebras.) As in the case of finite dimension, their result gives an easy approach to the Weyl–Kac character formula (see [4]). Three years later, J. Lepowsky discussed a complex defined by a lowest weight module. By following all the steps in [4], he proved Kostant's cohomology formula with the same assumptions as in [4] (see [5]).

The purpose of this paper is to investigate the relations between Kostant's cohomology and homology formulas and Weyl–Kac's character formula for Kac–Moody Lie algebras. The program follows what F. Aribaud did in 1967. Our investigations yield a proof of the homology and cohomology formulas in a more general case, i.e., that when the reductive part of the parabolic subalgebra is of arbitrary type. (In this case, we also an analogue of the resolution given in [4, Theorem 8.7] at the end of this paper.) But we still assume that A is symmetrizable.

The structure of this paper imitates [3], going from the character formula and denominator identities to the cohomology and homology formulas. In Section 4 we prove $H_j(\mathbf{u}^-, M(\lambda)) = H^j(\mathbf{u}^+, M(\lambda))$ (see Section 1), by using the Cartan involution, and then concentrating on $H_j(\mathbf{u}^-, M(\lambda))$. We use H. Garland and J. Lepowsky's method in the appendix of [4] and F. Aribaud's method in [3, Sect. 4] to prove that any irreducible component of $H_j(\mathbf{u}^-, M(\lambda))$ must be of the form $m(\mu)$ (see Section 1) with $|\mu+\rho|^2 = |\lambda+\rho|^2$; this result is one of the cornerstones of this paper.

The results of this paper strongly depend on [1, Corollary 10.7] By that theorem almost every module in this paper is completely reducible.

0. Notation

Let $A = (a_{ij})$ be an $n \times n$ symmetrizable generalized Cartan matrix, that is, $a_{ij} \in Z$ (the integers), for all i, j, satisfying (I) $a_{ii} = 2$, (II) $a_{ij} \leqslant 0$, $i \neq j$, (III) $a_{ij} = 0 \Rightarrow a_{ji} = 0$. We say A is symmetrizable if there is a matrix $D = \operatorname{diag}\{\varepsilon_1, ..., \varepsilon_n\}$ with all $\varepsilon_i > 0$, such that DA is a symmetric matrix.

Here, we follow all the definitions in [1, Chaps. 1–3]. Let $\mathbf{g} = \mathbf{g}(A)$ be the Kac–Moody Lie algebra of A over the complex number field C defined by the Cartan matrix A, $\mathbf{h}$ the Cartan subalgebra of $\mathbf{g}$ with $\dim \mathbf{h} = 2n -$ rank A, and f_i, $\alpha_i^\vee$, e_i for $i = 1, ..., n$ the Chevalley generators of $\mathbf{g}$, where $\mathbf{h}$ acts on $\mathbf{g}$ diagonally, i.e.,

$$\mathbf{g} = \mathbf{h} + \sum_{\alpha \in \mathbf{h}^*} \mathbf{g}^\alpha, \qquad \text{where} \quad \mathbf{g}^\alpha = \{x \in \mathbf{g} \mid [h, x] = \alpha(h)x, \ \forall h \in \mathbf{h}\}.$$

Let $\Delta = \{\alpha \in \mathbf{h} \mid \alpha \neq 0,\ \mathbf{g}^\alpha \neq 0\}$ be the root system of $\mathbf{g}$ and $\Pi = \{\alpha_1, ..., \alpha_n\} \subset \Delta$ be the root basis of Δ satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$, which gives an order in Δ. So $\Delta = \Delta^- \cup \Delta^+$, where $\Delta^+ = \{\alpha \in \Delta \mid \alpha = \sum_{i=1}^n k_i \alpha_i$, all $k_i \in Z_+\}$, $\Delta^- = -\Delta^+$. Then we can write

$$\mathbf{g} = \mathbf{n}^- + \mathbf{h} + \mathbf{n}^+, \qquad \text{where} \quad \mathbf{n}^\pm = \sum_{\alpha \in \Delta^\pm} \mathbf{g}^\alpha.$$

Since A is symmetrizable, there is a non-degenerate invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathbf{g}$, which is also non-degenerate on $\mathbf{h}$. It induces a linear isomorphism $v\colon \mathbf{h} \to \mathbf{h}^*$, and v induces a bilinear form on $\mathbf{h}^*$ which is also written $(\cdot, \cdot)$. Let r_i be the reflection on $\mathbf{h}^*$ determined by $\alpha_i \in \Pi$ for $i = 1, ..., n$, and W the Weyl group generated by $r_1, ..., r_n$. W also act on $\mathbf{h}$, if we identify r_i with $r_i^\vee$, where $r_i^\vee$ is the reflection determined by $\alpha_i^\vee \in \mathbf{h}$ for $i = 1, ..., n$.

Let S be an arbitrary subset of $\{1, ..., n\}$. For convenience, we simply write $S = \{1, ..., s\}$. Assume $\mathbf{r}$ is the subalgebra of $\mathbf{g}$ generated by $e_1, ..., e_s$, $f_1, ..., f_s$, and $\mathbf{h}$. If we denote the root system of $\mathbf{r}$ by Δ_S, then $\Delta_S = \Delta_S^- \cup \Delta_S^+$, where $\Delta_S^\pm = \Delta_S \cap \Delta^\pm$, and the root basis of Δ_S^+ is $\Pi_S = \{\alpha_1, ..., \alpha_s\}$. Then

$$\mathbf{r} = \mathbf{m}^- + \mathbf{h} + \mathbf{m}^+, \qquad \text{where} \quad \mathbf{m}^\pm = \sum_{\alpha \in \Delta_S^\pm} \mathbf{g}^\alpha.$$

We denote by W_S the Weyl group of $\mathbf{r}$, which is generated by $r_1, ..., r_s$. Let $\Delta^\pm(S) = \Delta^\pm \backslash \Delta_S^\pm$, $\mathbf{u}^\pm = \sum_{\alpha \in \Delta^\pm(S)} \mathbf{g}^\alpha$, $W(S) = \{w \in W \mid w\Delta^- \cap \Delta^+ \subseteq \Delta^+(S)\}$. Then $[\mathbf{r}, \mathbf{u}^\pm] \subseteq \mathbf{u}^\pm$, and $\mathbf{u}^+$, $\mathbf{u}^-$ are $\mathbf{r}$-modules. So $\Lambda^j \mathbf{u}^+$, $\Lambda^j \mathbf{u}^-$ are $\mathbf{r}$-modules, $j = 0, 1, 2, ...$. The algebra $\mathbf{r}$ acts on them by derivations. Let

$$P = \{\lambda \in \mathbf{h}^* \mid \lambda(\alpha_i^\vee) \text{ are non-negative integers, } i = 1, ..., n\}$$

$$P_S = \{\lambda \in \mathbf{h}^* \mid \lambda(\alpha_j^\vee) \text{ are non-negative integers, for } j \in S\}.$$

We choose an element ρ in P satisfying $\rho(\alpha_i^\vee) = 1$, $i = 1, ..., n$.

For $\lambda \in P$ (or $\in P_S$), we denote the irreducible highest weight $\mathbf{g}$ (or $\mathbf{r}$)-module with highest weight λ by $M(\lambda)$ (or $m(\lambda)$). Then $\hom(\Lambda^j \mathbf{u}^+, M(\lambda))$ and $\Lambda^j \mathbf{u}^- \otimes M(\lambda)$ are $\mathbf{r}$-modules in the usual way. (Note: When $\Lambda^j \mathbf{u}^+$ is finite-dimensional, $\hom(\Lambda^j \mathbf{u}^+, M(\lambda))$ is an $\mathbf{h}$-weight module. When $\Lambda^j \mathbf{u}^+$ is infinite-dimensional, we still want $\hom(\Lambda^j \mathbf{u}^+, M(\lambda))$ to be an $\mathbf{h}$-weight module. So we have to assume further conditions on the definition of hom(,), see Section 4.) Now consider the following $\mathbf{r}$-module complex and cocomplex:

(1)

$$\cdots \longrightarrow \Lambda^j \mathbf{u}^- \otimes M(\lambda) \xrightarrow{d_j} \Lambda^{j-1}\mathbf{u}^- \otimes M(\lambda) \longrightarrow \cdots \longrightarrow \Lambda^0 \mathbf{u}^- \otimes M(\lambda) \longrightarrow 0$$

$$d_j(c_1 \wedge \cdots \wedge c_j \otimes w) = \sum_{i=1}^{j} (-1)^i c_1 \wedge \cdots \wedge \hat{c}_i \wedge \cdots \wedge c_j \otimes c_i w + \sum_{r<t} (-1)^{r+t} [c_r, c_t] \wedge c_1 \wedge \cdots \wedge \hat{c}_r \wedge \cdots \wedge \hat{c}_t \wedge \cdots \wedge c_j \otimes w$$

for $c_1, ..., c_j \in \mathbf{u}^-$, $w \in M(\lambda)$.

(2)

$$\cdots \longleftarrow \hom(\Lambda^{j+1}\mathbf{u}^+, M(\lambda)) \xleftarrow{d_j} \hom(\Lambda^j \mathbf{u}^+, M(\lambda)) \longleftarrow \cdots \longleftarrow \hom(\Lambda^0 \mathbf{u}^+, M(\lambda)) \longleftarrow 0$$

$$(d^j f)(c_1 \wedge \cdots \wedge c_{j+1}) = \sum_{i=1}^{j+1} (-1)^{i+1} c_i f(c_1 \wedge \cdots \wedge \hat{c}_i \wedge \cdots \wedge c_{j+1}) + \sum_{r<t} (-1)^{r+t} f([c_r, c_t] \wedge c_1 \wedge \cdots \wedge \hat{c}_r \wedge \cdots \wedge \hat{c}_t \wedge \cdots \wedge c_{j+1})$$

for $c_1, ..., c_{j+1} \in \mathbf{u}^+$, $f \in \hom(\Lambda^j \mathbf{u}^+, M(\lambda))$.

The homology groups of these sequences are also $\mathbf{r}$-modules, denoted as $H_*(\mathbf{u}^-, M(\lambda))$ and $H^*(\mathbf{u}^+, M(\lambda))$. We are going to show that

$$H_j(\mathbf{u}^-, M(\lambda)) = H^j(\mathbf{u}^+, M(\lambda)) = \sum_{w \in W(S),\, l(w)=j} m(w(\lambda+\rho)-\rho).$$

This relation is called Kostant's homology and cohomology formula. Originally, it was obtained by B. Kostant in 1961 in the case of finite dimensional algebras and modules.

1. The Grothendieck Ring of the Lie Algebra r

The main work of this section is to define the Grothendieck ring, a generalization of the ring of characters.

LEMMA 1. *Let R be a non-empty set, and assume that R has an addition and a multiplication satisfying the following conditions*:

(1) *Both addition and multiplication are commutative and associative.*

(2) *R has a zero* 0, *and a unit* 1.

(3) $x+y=x+z$, *implies* $y=z$, *for any*, x, y, $z \in R$

(4) $(x+y)z=xy+xz$, *for any* x, y, $z \in R$.

Then there exists a unique ring $\mathscr{R} \supseteq R$, *such that for any* $x \in \mathscr{R}$, *there exist* $y, z \in R$ *with* $y-z=x$.

Proof. Same as that we use to obtain the integers from the natural numbers. Q.E.D.

We call an $\mathbf{h}$-module M a weight module if

$$M=\bigoplus_{\mu \in \mathbf{h}^*} M_\mu, \qquad \text{where } M_\mu=\{m \in M \mid hm=\mu(h)m, \text{ any } h \in \mathbf{h}\}$$

if $M_\mu \neq 0$ we say μ is a weight of M, and M_μ is the weight space of μ.

Let $v \in \mathbf{h}^*$; define $D(v)=\{v-\sum_{i=1}^n k_i\alpha_i \mid k_1, ..., k_n$ are nonnegative integers$\}$. Let $\mathcal{O}$ be the category of all $\mathbf{h}$-weight modules M, such that (1) There exists $v_1, ..., v_t \in \mathbf{h}^*$, such that the weight set of M is contained in $\bigcup_{i=1}^t D(v_i)$. (2) Every weight space of M is finite dimensional.

Let $C(\mathbf{r})$ be the category of all $\mathbf{r}$-modules M, satisfying: (a) As an $\mathbf{h}$-module, $M \in \mathcal{O}$; (b) M is a completely reducible $\mathbf{r}$-module, and every irreducible component of M is of the form $m(\mu)$ with $\mu \in P_S$.

Note. If $M \in C(\mathbf{r})$, and $m(\mu)$ is a component of M, then M has only finitely many components isomorphic to $m(\mu)$, by condition (a).

Let A_S be the submatrix of A corresponding to S. There exists a $2s-\operatorname{rank} A_S$ dimensional subspace $\mathbf{h}_S$ of $\mathbf{h}$ with $\alpha_1^\vee, ..., \alpha_s^\vee \in \mathbf{h}_S$, and $(\cdot,\cdot)$ is non-degenerate on it. Then $\mathbf{h}=\mathbf{h}_S \oplus \mathbf{h}(S)$, where $\mathbf{h}(S)=\mathbf{h}_S^\perp$, $\alpha_1, ..., \alpha_s$ vanish on $\mathbf{h}(S)$. And $\mathbf{g}_S=\mathbf{n}_S^- +\mathbf{h}_S+\mathbf{n}_S^+$ is the Kac–Moody Lie algebra of A_S. Then

$$\mathbf{r}=\mathbf{g}_S \oplus \mathbf{h}(S), \qquad [\mathbf{h}(S), \mathbf{g}_S]=0.$$

LEMMA 2. *$M(\lambda)$ and $\Lambda\mathbf{u}^- \otimes M(\lambda)$ are in $C(\mathbf{r})$, for $\lambda \in P$; and $C(\mathbf{r})$ is closed under tensor products.*

Proof. Let $\mu=\lambda-\sum_{ii=s+1}^n m_i\alpha_i \in \mathbf{h}^*$, where all m_i are nonnegative integers, and consider the $\mathbf{r}$-submodule of $M(\lambda)$,

$$M(\lambda)^\mu=\sum_{k_1, ..., k_s \in Z_+} M(\lambda)_{\mu-(k_1\alpha_1+\cdots+k_s\alpha_s)}.$$

Since $\alpha_1, ..., \alpha_s$ are independent on $\mathbf{h}_S$, the $\mathbf{h}_S$-weight spaces of $M(\lambda)^\mu$ are $\mathbf{h}$-weight spaces, so every $\mathbf{h}_S$-weight space is finite dimensional. By [1, Corollary 10.7], $M(\lambda)^\mu$ is a completely reducible $\mathbf{g}_S$-module, and any $h \in \mathbf{h}(S)$ acts on $M(\lambda)^\mu$ by a constant $\mu(h)$. Hence, $M(\lambda)^\mu$ is in the category $C(\mathbf{r})$, and so is $M(\lambda)$. Similarly we can prove that $\Lambda \mathbf{u}^- \otimes M(\lambda) \in C(\mathbf{r})$. Still by [1, Corollary 10.7], $m(\mu) \otimes m(\nu)$ is a completely reducible $\mathbf{g}_S$-module, for $\mu, \nu \in P_S$. And any $h \in \mathbf{h}(S)$ acts on $m(\mu) \otimes m(\nu)$ by a constant $(\mu + \nu)(h)$, hence $m(\mu) \otimes m(\nu) \in C(\mathbf{r})$. This implies $C(\mathbf{r})$ is closed under tensor products. Q.E.D.

For any $M \in C(\mathbf{r})$, let $ch_\mathbf{r} M$ be the class of all $N \in C(\mathbf{r})$ such that $N \cong M$. And let $R(\mathbf{r}) = \{ch_\mathbf{r} M \mid M \in C(\mathbf{r})\}$.

Now we introduce addition and multiplication in $R(\mathbf{r})$: if $M_i \in C(\mathbf{r})$, for $i \in I$, define

$$\sum_{i \in I} ch_\mathbf{r} M_i = ch_\mathbf{r} \left(\bigoplus_{i \in I} M_i \right); \qquad \text{if } I \text{ is finite, or } \bigoplus_{i \in I} M_i \in C(\mathbf{r}).$$

For M, $N \in C(\mathbf{r})$, define

$$ch_\mathbf{r} M ch_\mathbf{r} N = ch_\mathbf{r}(M \otimes N).$$

Then it is easy to see that both addition and multiplication are commutative and associative, the zero module is the zero of $R(\mathbf{r})$, the one dimensional trivial module is the unit element of $R(\mathbf{r})$, and multiplciation distributes over addition. By the Note on the category $C(\mathbf{r})$, $ch_\mathbf{r} P + ch_\mathbf{r} Q = ch_\mathbf{r} P + ch_\mathbf{r} R$ implies $ch_\mathbf{r} Q = ch_\mathbf{r} R$. These are all the conditions in Lemma 1, so we obtain a ring $\mathscr{R}(\mathbf{r})$. And $R(\mathbf{r})$ has other properties,

$$\sum_{i \in I} \sum_{j \in J} ch_\mathbf{r} M_{ij} = \sum_{j \in J} \sum_{i \in I} ch_\mathbf{r} M_{ij}, \qquad \text{if } \bigoplus_{i \in I, j \in J} M_{ij} \in C(\mathbf{r});$$

$$\sum_{j \in J} (ch_\mathbf{r} M_j ch_\mathbf{r} N) = \left(\sum_{j \in J} ch_\mathbf{r} M_j \right) ch_\mathbf{r} N, \qquad \text{if } \bigoplus_{j \in J} M_j,\ N \in C(\mathbf{r}).$$

For $S = \{1, ..., n\}$ we obtain the category $C(\mathbf{g})$ and the ring $\mathscr{R}(\mathbf{g})$. For $S = \varnothing$ we obtain the ring $\mathscr{R}(\mathbf{h})$. In the latter case all the elements in $\mathscr{R}(\mathbf{h})$ are characters, so we write $ch_\mathbf{r} m(\mu)$ simply by $e(\mu)$ as is traditional.

By Lemma 2, we have the following homomorphisms:

$$\begin{aligned} \operatorname{Res}(\mathbf{g}, \mathbf{r})&: \mathscr{R}(\mathbf{g}) \to \mathscr{R}(\mathbf{r}), & ch_\mathbf{g} M &\mapsto ch_\mathbf{r} M; \\ \operatorname{Res}(\mathbf{r}, \mathbf{h})&: \mathscr{R}(\mathbf{r}) \to \mathscr{R}(\mathbf{h}), & ch_\mathbf{r} M &\mapsto ch_\mathbf{h} M; \\ \operatorname{Res}(\mathbf{g}, \mathbf{h})&: \mathscr{R}(\mathbf{g}) \to \mathscr{R}(\mathbf{h}), & ch_\mathbf{g} M &\mapsto ch_\mathbf{h} M. \end{aligned}$$

LEMMA 3. $\operatorname{Res}(\mathbf{g}, \mathbf{r})$, $\operatorname{Res}(\mathbf{r}, \mathbf{h})$, $\operatorname{Res}(\mathbf{g}, \mathbf{h})$ *are* 1–1 *maps.*

Proof. If $0 \neq x \in \operatorname{kernel} \operatorname{Res}(\mathbf{r}, \mathbf{h})$, choose P, $Q \in C(\mathbf{r})$ such that they do not have any common components, $ch_{\mathbf{r}} P - ch_{\mathbf{r}} Q = x$. Assume there are $v_1, ..., v_l \in \mathbf{h}^*$, such that the weights of P and Q are contained in $\bigcup_{i=1}^{l} D(v_i)$. We can assume $v_i - v_j$ is not contained in the group generated by $\alpha_1, ..., \alpha_n$, for any $i \neq j$. Let $\mu = v_k - (m_1\alpha_1 + \cdots + m_n\alpha_n)$ be a weight of P or Q such that $m_1 + \cdots + m_n$ is minimal. We can assume that μ is a weight of P. Then μ must be the highest weight of a component of P. By the choice of P, Q, and μ, μ is not a weight of Q. This implies $\mathrm{Res}(\mathbf{r}, \mathbf{h})(x) \neq 0$, contradicting the choice of x. Hence, $\mathrm{Res}(\mathbf{r}, \mathbf{h})$ is a 1–1 map. $\mathbf{g}$ is a special case of $\mathbf{r}$, hence $\mathrm{Res}(\mathbf{g}, \mathbf{h})$ is a 1–1 map. $\mathrm{Res}(\mathbf{g}, \mathbf{h}) = \mathrm{Res}(\mathbf{r}, \mathbf{h})\ \mathrm{Res}(\mathbf{g}, \mathbf{r})$ implies $\mathrm{Res}(\mathbf{g}, \mathbf{r})$ is a 1–1 map. Q.E.D.

2. Weyl Groups

Recall that W is the Weyl group of the root system Δ, the subgroup W_S of W is the Weyl groups of the root system Δ_S. For $w \in W$, define $\Phi_w = w\Delta^- \cap \Delta^+$,

$$n(w) \overset{\text{def}}{=} \text{the number of elements in } \Phi_w$$

$$l(w) \overset{\text{def}}{=} \text{length of } w \text{, i.e., the smallest number } m \text{ such that}$$

$$w = r_{i_1} \cdots r_{i_m},\ i_j \in \{1, ..., n\}.$$

Choose an element $\rho \in \mathbf{h}^*$ such that $\rho(\alpha_i^\vee) = 1$, $i = 1, ..., n$.

Lemma 4 (This result is established in [7, Sect. 2]). (1) *If* $\alpha_j \notin \Phi_w$, *then* $\Phi_{r_j w} = r_j \Phi_w \cup \{\alpha_j\}$.

(2) $n(w) = l(w)$, *any* $w \in W$.

(3) $\rho - w(\rho) = \sum_{\alpha \in \Phi_w} \alpha$.

(*Note.* It is still true even if A is not symmetrizable, see [1, Chap. 3].)

Proof. See [7, Sect. 2]. Q.E.D.

In general, $l(r_j w) = l(w) \pm 1$. From the proof of this lemma, we could see that $+$ holds if and only if $\alpha_j \notin \Phi_w$. This tells us how to compute the set Φ_w, if $l(w) = j$, $w = r_{i_1} \cdots r_{i_j}$, then $\Phi_w = \{\alpha_{i_1}, r_{i_1}(\alpha_{i_2}), ..., r_{i_1} \cdots r_{i_{j-1}}(\alpha_{i_j})\}$. It follows that, if $l(w_1 w_2) = l(w_1) + l(w_2)$, then $\Phi_{w_1 w_2} = w_1 \Phi_{w_2} \cup \Phi_{w_1}$, which generalizes (1). And (3) says $\Phi_{w_1} = \Phi_{w_2}$ if and only if

$$\sum_{\alpha \in \Phi_{w_1}} = \sum_{\alpha \in \Phi_{w_2}} \qquad \text{if and only if } w_1 = w_2.$$

LEMMA 5. *If Δ is a root system, and if $\Sigma \subseteq \Delta$ satisfies:*

(1) *Δ is a disjoint union of Σ and $-\Sigma$.*

(2) *Σ is closed. (If α, $\beta \in \Sigma$ with $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Sigma$.)*

(3) *$\Delta^+ \cap (-\Sigma)$ is a finite set.*

Then there exists $u \in W$ such that $u\Delta^+ = \Sigma$.

Proof. Assume $\Delta^+ \cap (-\Sigma) \neq \emptyset$. Then there is a simple root $\alpha_{i_1} \in \Delta^+ \cap (-\Sigma)$, by the conditions (1), (2). We know that $r_{i_1}\Sigma$ also satisfies conditions (1), (2), and

$$\begin{aligned}\Delta^+ \cap (-r_{i_1}(\Sigma)) &= r_{i_1}(r_{i_1}\Delta^+ \cap (-\Sigma)) \\ &= r_{i_1}(((\Delta^+ \setminus \{\alpha_{i_1}\}) \cup \{-\alpha_{i_1}\}) \cap (-\Sigma)) \\ &= r_{i_1}((\Delta^+ \setminus \{\alpha_{i_1}\}) \cap (-\Sigma)) \\ &= r_{i_1}(\Delta^+ \cap (-\Sigma) \setminus \{\alpha_{i_1}\}).\end{aligned}$$

Hence the number of elements in $\Delta^+ \cap (-r_{i_1}\Sigma)$ is less than the number of elements in $\Delta^+ \cap (-\Sigma)$. By induction we find a $w = r_{i_l} \cdots r_{i_1} \in W$, such that $\Delta^+ \cap (-w\Sigma) = \emptyset$. Then $\Delta^+ \subseteq w\Sigma$, $\Delta^- = -\Delta^+ \subseteq -w\Sigma$, so $\Delta^+ = w\Sigma$. For $u = w^{-1}$, we have $u\Delta^+ = \Sigma$. Q.E.D.

By this lemma, it is easy to see that: if Φ is a finite subset of Δ^+, such that Φ and $\Delta^+ \setminus \Phi$ are closed, then there is a $w \in W$ with $\Phi_w = \Phi$, and the map $w \mapsto \Phi_w$ is a bijection of the Weyl group onto the family of all such Φ. So, [2, Proposition 5.10] is still valid in our case.

Recall that $W(S) = \{w \in W \mid w\Delta^- \cap \Delta^+ = \Phi_w \subseteq \Delta^+(S)\}$. Next, we generalize [2, Proposition 5.13]. This result does not seem very important in [2]. But here, we depend on it.

PROPOSITION 6. *$W(S)$ is a set of representatives for the right coset space W/W_S.*

Proof. Let $t \in W$, $F_1 = t\Delta^- \cap \Delta_S^+$, $F_2 = t\Delta^+ \cap \Delta_S^+$, $\Sigma = (-F_1) \cup F_2$. We show that $t = un$, where $u \in W_S$, $n \in W(S)$.

It is clear that $\Sigma \cup (-\Sigma) = \Delta_S$. Choose $h \in \mathbf{h}$ with $\alpha(h) > 0$, for all $\alpha \in \Delta^+$. Then for any $\beta \in \Sigma$, $\beta(t(h)) = t^{-1}(\beta)(h) > 0$. This means that $\Sigma \subseteq \Delta_S$ satisfies all conditions in Lemma 5. Hence, there exists a $u \in W_S$ such that $u\Delta_S^+ = \Sigma$. (The following proof can be seen in [3, Proposition 4].) Let $n = u^{-1}t$, $t = un$; we check that $n \in W(S)$.

$$u\Delta_S^- \cap \Delta_S^+ = (-\Sigma) \cap \Delta_S^+ = F_1 = t\Delta^- \cap \Delta_S^+$$

$$n\Delta^- \cap u^{-1}\Delta_S^+ = u^{-1}(t\Delta^- \cap \Delta_S^+) = u^{-1}(u\Delta_S^- \cap \Delta_S^+) = \Delta_S^- \cap u^{-1}\Delta_S^+$$

$$\Rightarrow n\Delta^- \cap (u^{-1}\Delta_S^+ \cap \Delta_S^+) = \emptyset,$$

and

$$u^{-1}\Delta_S^- \cap \Delta_S^+ = u^{-1}(\Delta_S^- \cap u\Delta_S^+) = -u^{-1}(\Delta_S^+ \cap u\Delta_S^-)$$
$$= -u^{-1}(t\Delta^- \cap \Delta_S^+) = n\Delta^+ \cap u^{-1}\Delta_S^-$$
$$\Rightarrow n\Delta^- \cap (u^{-1}\Delta_S^- \cap \Delta_S^+) = \varnothing.$$

Hence, we obtain

$$n\Delta^- \cap \Delta_S^+ = \varnothing \Rightarrow n\Delta^- \cap \Delta^+ \subseteq \Delta^+(S), \qquad n \in W(S).$$

Next let n_1, $n_2 \in W(S)$, such that $W_S n_1 = W_S n_2$. Then $n_1 n_2^{-1} = u \in W_S$.

$$n_1 n_2^{-1}\Delta_S^+ = u\Delta_S^+ \subseteq \Delta_S.$$

Since $n_2 \in W(S) \Rightarrow n_2\Delta^- \cap \Delta^+ \subseteq \Delta^+(S) \Rightarrow n_2\Delta^+ \cap \Delta^+ \supseteq \Delta_S^+$, $n_2^{-1}\Delta_S^+ \subseteq \Delta^+$.

$$n_1 n_2^{-1}\Delta_S^+ \subseteq n_1\Delta^+ = (n_1\Delta^+ \cap \Delta^+) \cup (n_1\Delta^+ \cap \Delta^-)$$
$$n_1\Delta^+ \cap \Delta^- = -(n_1\Delta^- \cap \Delta^+) \subseteq -\Delta^+(S)$$
$$n_1 n_2^{-1}\Delta_S^+ \subseteq \Delta_S \cap n_1\Delta^+ = \Delta_S \cap ((n_1\Delta^+ \cap \Delta^+) \cup (n_1\Delta^+ \cap \Delta^-))$$
$$= \Delta_S \cap (n_1\Delta^+ \cap \Delta^+) \subseteq \Delta_S^+,$$

i.e., $u\Delta_S^+ \subseteq \Delta_S^+$. Since $u \in W_S \Rightarrow u\Delta^+(S) \subseteq \Delta^+(S) \Rightarrow u\Delta^+ \subseteq \Delta^+$, by Lemma 4 $n(u)(=l(u)=0$, $u=1$, and $n_1 = n_2$. Q.E.D.

$l(un) = l(u) + l(n)$, for $u \in W_S$, $n \in W(S)$. In fact,

$$un(\Delta^- \cap \Delta^+) = u((n\Delta^- \cap \Delta^+) \cup (n\Delta^- \cap \Delta^-)) \cap \Delta^+$$
$$= (u\Phi_n \cap \Delta^+) \cup (u(n\Delta^- \cap \Delta^-) \cap \Delta^+).$$

It is clear that $u\Phi_n \subset \Delta^+(S)$. $n \in W(S)$ implies $\Delta_S^- \subset n\Delta^-$, and $\Phi_u = u\Delta^- \cap \Delta^+ = u\Delta_S^- \cap \Delta_S^+$. These imply $u(n\Delta^- \cap \Delta^-) \cap \Delta^+ = \Phi_u \subset \Delta_S^+$. So, Φ_{un} is a disjoint union of the sets $u\Phi_n$ and Φ_u. For any $n \in W(S)$, n is the element in the coset $W_S n$ with the shortest length.

Note. In general, $W(S)$ is not a subgroup of W. This will happen, if and only if the subsets S and its complement are disconnected.

3. Corollary of Weyl's Character Formula

For $\lambda \in P$ (or P_S), we write the irreducible highest weight $\mathbf{g}$ (or $\mathbf{r}$)-module with the highest weight λ by $M(\lambda)$ (or $m(\lambda)$). We will use

Kac–Weyl's character formula and denominator identities of $M(\lambda)$, $m(\lambda)$ written as

$$ch_{\mathbf{h}} M(\lambda) = \frac{\sum_{w \in W} \det(w)\, e(w(\lambda+\rho)-\rho)}{\sum_{w \in W} \det(w)\, e(w(\rho)-\rho)}$$

$$ch_{\mathbf{h}} m(\lambda) = \frac{\sum_{w \in W_S} \det(w)\, e(w(\lambda+\rho)-\rho)}{\sum_{w \in W_S} \det(w)\, e(w(\rho)-\rho)}$$

$$e(\rho) \prod_{\alpha \in \Delta^+} (1-e(-\alpha))^{\dim \mathbf{g}^\alpha} = \sum_{v \in W} \det(v)\, e(v(\rho))$$

$$e(\rho) \prod_{\alpha \in \Delta_S^+} (1-e(-\alpha))^{\dim \mathbf{g}^\alpha} = \sum_{u \in W_S} \det(u)\, e(u(\rho)).$$

Note. We know that the central subalgebra $\mathbf{h}(S)$ of $\mathbf{r}$ is killed by the root basis $\{\alpha_1, ..., \alpha_s\}$ of $\mathbf{r}$. Therefore, the restrictions of $\alpha_1, ..., \alpha_s$ to the subalgebra $\mathbf{h}_S$ of $\mathbf{h}$ are still linearly independent; and x acts on the $\mathbf{r}$-module $m(\lambda)$ by the constant $\lambda(x)$, for any $x \in \mathbf{h}(S)$. Therefore, $m(\lambda)$ as $\mathbf{g}_S$-module is irreducible. Combining all of this and the Weyl–Kac character formula for the algebra $\mathbf{g}_S$, we could say that the second formula is valid. Similarly the 4th one is valid.

Using these formulas, we have:

PROPOSITION 7 (Weyl–Kostant). *For $\lambda \in P$,*

$$ch_{\mathbf{r}} M(\lambda) = \frac{\sum_{w \in W(S)} \det(w)\, ch_{\mathbf{r}} m(w(\lambda+\rho)-\rho)}{\sum_{w \in W(S)} \det(w)\, ch_{\mathbf{r}} m(w(\rho)-\rho)}.$$

Proof. For any $w \in W(S)$, $i \in S$, $w^{-1}(\alpha_i)$ is a positive root, $(w(\lambda+\rho), \alpha_i) = (\lambda+\rho, w^{-1}(\alpha_i)) > 0$, $\Rightarrow (w(\lambda+\rho)-\rho, \alpha_i) \geqslant 0$, $\Rightarrow w(\lambda+\rho)-\rho \in P_S$.

We have

$$ch_{\mathbf{h}} m(w(\lambda+\rho)-\rho) = \frac{\sum_{u \in W_S} \det(u)\, e(u(w(\lambda+\rho)-\rho+\rho)-\rho)}{\sum_{u \in W_S} \det(u)\, e(u(\rho)-\rho)} = \frac{\sum_{u \in W_S} \det(u)\, e(uw(\lambda+\rho))}{\sum_{u \in W_S} \det(u)\, e(u(\rho))}.$$

In case $\lambda = 0$ we have

$$ch_{\mathbf{h}} m(w(\rho)-\rho) = \frac{\sum_{u \in W_S} \det(u)\, e(uw(\rho))}{\sum_{u \in W_S} \det(u)\, e(u(\rho))}.$$

Now substitute these two identities into the right side in the formula of this proposition. We obtain

$$\frac{\sum_{w\in W(S)}\sum_{u\in W_S}\det(uw)\,e(uw(\lambda+\rho))}{\sum_{w\in W(S)}\sum_{u\in W_S}\det(uw)\,e(uw(\rho))}.$$

By Proposition 5 this is equal to the right side of the Weyl–Kac formula. In other words, when we apply $\operatorname{Res}(\mathbf{r},\mathbf{h})$ in both sides of the formula in this proposition we obtain the Weyl–Kac character formula. By Lemma 3, $\operatorname{Res}(\mathbf{r},\mathbf{h})$ is a 1–1 map, and this proves the formula. Q.E.D.

COROLLARY 8. *For* $\lambda\in P$,

$$\sum_{j=0}^{\infty}(-1)^j\,ch_{\mathbf{r}}(H_j(\mathbf{u}^-,M(\lambda)))=\sum_{w\in W(S)}\det(w)\,ch_{\mathbf{r}}m(w(\lambda+\rho)-\rho).$$

Proof. By Euler's formula, we have

$$\begin{aligned}\sum_{j=0}^{\infty}(-1)^j\,ch_{\mathbf{r}}H_j(\mathbf{u}^-,M(\lambda))&=\sum_{j=0}^{\infty}(-1)^j\,ch_{\mathbf{r}}(\Lambda^j\mathbf{u}^-\otimes M(\lambda))\\&=\left(\sum_{j=0}^{\infty}(-1)^j\,ch_{\mathbf{r}}\Lambda^j\mathbf{u}^-\right)ch_{\mathbf{r}}M(\lambda).\end{aligned}$$

By Proposition 7 we need only to prove

$$\sum_{j=0}^{\infty}(-1)^j\,ch_{\mathbf{r}}\Lambda^j\mathbf{u}^-=\sum_{w\in W(S)}\det(w)\,ch_{\mathbf{r}}m(w(\rho)-\rho).$$

We apply $\operatorname{Res}(\mathbf{r},\mathbf{h})$ on both sides of it. By Weyl–Kac' denominator identity, we have

$$\begin{aligned}\text{left}&=\sum_{j=1}^{\infty}(-1)^j\,ch_{\mathbf{h}}\Lambda^j\mathbf{u}^-=\prod_{\alpha\in\Delta^+(S)}(1-e(-\alpha))^{\dim\mathbf{g}^\alpha}\\&=\frac{e(\rho)\prod_{\alpha\in\Delta^+}(1-e(-\alpha))^{\dim\mathbf{g}^\alpha}}{e(\rho)\prod_{\alpha\in\Delta_S^+}(1-e(-\alpha))^{\dim\mathbf{g}^\alpha}}\\&=\frac{\sum_{v\in W}\det(v)\,e(v(\rho))}{\sum_{u\in W_S}\det(u)\,e(u(\rho))}.\end{aligned}$$

By Weyl–Kac' character formula, and Proposition 7 we have

$$\begin{aligned}\text{right} &= \sum_{w \in W(S)} \det(w)\, ch_{\mathbf{h}}(m(w(\rho)-\rho)) \\ &= \sum_{w \in W(S)} \det(w) \frac{\sum_{u \in W_S} \det(u)\, e(uw(\rho)-\rho)}{\sum_{u \in W_S} \det(u)\, e(u(\rho)-\rho)} \\ &= \frac{\sum_{v \in W} \det(v)\, e(v(\rho))}{\sum_{u \in W_S} \det(u)\, e(u(\rho))}.\end{aligned}$$

Now left = right. Recalling that $\operatorname{Res}(\mathbf{r}, \mathbf{h})$ is a 1–1 map, we obtain the formula. Q.E.D.

This formula means that for any $w \in W(S)$, there exists a j, such that $m(w(\lambda+\rho)-\rho)$ is a component of $H_j(\mathbf{u}^-, M(\lambda))$. In Section 5, we see these are the only components that $H_j(\mathbf{u}^-, M(\lambda))$ could have.

4. The Complex $\{\Lambda^*\mathbf{u}^- \otimes M(\lambda), d_*\}$ and the Cocomplex $\{\hom(\Lambda^*\mathbf{u}^+, M(\lambda)), d^*\}$

Homology and cohomology formulas were treated separately in [4, 5], but their structures are exactly the same. This suggests that they could be treated together. In this section, we will see that the complex (1) and the cocomplex (2) are related by the Cartan involution and several commutative diagrams.

If M, N are two $\mathbf{h}$-weight modules, and every weight space is finite dimensional, let

$$\hom(M, N) \overset{\text{def}}{=} \{f \mid f \text{ is a linear map from } M \text{ to } N,\ f(M_\alpha)=0 \text{ for all but finitely many weights } \alpha \text{ of } M.\}$$

Then $\hom(M, N)$ is an $\mathbf{h}$-weight module whose weight spaces could be infinite dimensional. But when $M = \Lambda^j\mathbf{u}^+$, $N = M(\lambda)$, all the weight spaces are finite dimensional. The complex field C is a trivial module. We write M^* for $\hom(M, C)$.

We are going to discuss relations between

$$\text{complex (1) } \{\Lambda^*\mathbf{u}^- \otimes M(\lambda), d_*\}$$

and

$$\text{cocomplex (2) } \{\hom(\Lambda^*\mathbf{u}^+, M(\lambda)), d^*\}$$

(see Section 1 for definition of d_* and d^*.)

First we introduce the $\mathbf{r}$-module complex,

$$\cdots \longrightarrow \Lambda^j \mathbf{u}^+ \otimes M(\lambda)^* \xrightarrow{\partial_j} \cdots \longrightarrow \Lambda^0 \mathbf{u}^+ M(\lambda)^* \xrightarrow{\partial_0} 0, \tag{3}$$

where the boundary maps ∂_* are defined as for complex (1). This complex induces an $\mathbf{r}$-module cocomplex,

$$\{\hom(\Lambda^* \mathbf{u}^+ \otimes M(\lambda)^*, C), \partial^*\}, \tag{4}$$

where $\partial^j(f) = f \circ \partial_{j+1}$, for $f \in \hom(\Lambda^j \mathbf{u}^+ \otimes M(\lambda)^*, C)$.

LEMMA 9. *As* $\mathbf{r}$*-modules,*

$$\frac{\text{kernel } \partial^j}{\text{image } \partial^{j-1}} \cong \left(\frac{\text{kernel } \partial_j}{\text{image } \partial_{j+1}}\right)^*.$$

Proof. For $f \in \text{kernel } \partial^j$, define $f' \in (\text{kernel } \partial_j / \text{image } \partial_{j+1})^*$, by $f'(a + \text{image } \partial_{j+1}) = f(a)$, $a \in \text{kernel } \partial_j$. Then $f \mapsto f'$, is a linear map.

If $f' = 0$, i.e., kernel $\partial_j \subset$ kernel f, let $\Lambda^{j+1} \mathbf{u}^+ \otimes M(\lambda)^* = (\text{image } \partial_j) \oplus Q$, where Q is an $\mathbf{h}$-submodule of $\Lambda^{j-1} \mathbf{u}^+ \otimes M(\lambda)^*$. Then there exists a linear map $f_1 \colon \Lambda^{j-1} \mathbf{u}^+ \otimes M(\lambda)^* \to C$, such that $f = f_1 \circ \partial_j$, and $f_1(Q) = 0$, hence $f_1 \in \hom(\Lambda^{j-1} \mathbf{u}^+ \otimes M(\lambda)^*, C)$, and $f \in \text{image } \partial^{j-1}$. On the other hand, for $f \in \text{image } \partial^{j-1}$, $f' = 0$.

If $g \in (\text{kernel } \partial_j / \text{image } \partial_{j+1})^*$, let $\Lambda^j \mathbf{u}^+ \otimes M(\lambda)^* = \text{kernel } \partial_j \oplus T$, where T is an $\mathbf{h}$-submodule of $\Lambda^j \mathbf{u}^+ \otimes M(\lambda)^*$. Then there exists a linear map $e \colon \Lambda^j \mathbf{u}^+ \otimes M(\lambda)^* \to C$ such that $e(a) = g(a + \text{image } \partial_{j+1})$, $e(T) = 0$, hence $e \in \hom(\Lambda^j \mathbf{u}^+ \otimes M(\lambda)^*, C)$, and $\partial^j(e) = e \circ \partial_{j+1} = 0$, $e' = g$.

Finally, it is easy to see that $(xf)' = xf'$, for $x \in \mathbf{r}$, $f \in \text{kernel } \partial^j$. Hence kernel ∂^j/image ∂^{j-1} is isomorphic to (kernel ∂_j/image ∂_{j+1})* as $\mathbf{r}$-modules.

Q.E.D.

It is well known that there are linear isomorphisms,

$$\Phi_j \colon \hom(\Lambda^j \mathbf{u}^+, M(\lambda)) \to \hom(\Lambda^j \mathbf{u}^+ \otimes M(\lambda)^*, C)$$

such that $\Phi_j(f)(u \otimes g) = g(f(u))$ for $f \in \hom(\Lambda^j \mathbf{u}^+, M(\lambda))$, $u \in \Lambda^j \mathbf{u}^+$, $g \in M(\lambda)^*$.

LEMMA 10. Φ_j *are* $\mathbf{r}$*-module homomorphisms, and*

$$\begin{array}{ccc}
\hom(\Lambda^{j+1} \mathbf{u}^+, M(\lambda)) & \xleftarrow{d^j} & \hom(\Lambda^j \mathbf{u}^+, M(\lambda)) \\
\Big\downarrow{\scriptstyle \Phi_{j+1}} & & \Big\downarrow{\scriptstyle \Phi_j} \\
\hom(\Lambda^{j+1} \mathbf{u}^+ \otimes M(\lambda)^*, C) & \xleftarrow[\partial^j]{} & \hom(\Lambda^j \mathbf{u}^+ \otimes M(\lambda)^*, C)
\end{array}$$

is commutative.

Proof. For $x \in \mathbf{r}$, $f \in \hom(\Lambda^j \mathbf{u}^+, M(\lambda))$, $u \in \Lambda^j \mathbf{u}^+$, $g \in M(\lambda)^*$.

$$\begin{aligned}
\Phi_j(xf)(u \otimes g) &= g((xf)(u)) = g(x(f(u)) - f(xu)) \\
(x\Phi(f))(u \otimes g) &= -\Phi(f)(x(u \otimes g)) = -\Phi(f)(xu \otimes g + u \otimes xg) \\
&= -g(f(xu)) - (xg)(f(u)) = g(-f(xu) + x(f(u))).
\end{aligned}$$

Hence Φ_j is an $\mathbf{r}$-homomorphism.

For $u_1, ..., u_{j+1} \in \mathbf{u}^+$, f, g as before,

$$\begin{aligned}
&\Phi_{j+1}(d^j(f))(u_1 \wedge \cdots \wedge u_{j+1} \otimes g) \\
&= g(d^j(f)(u_1 \wedge \cdots \wedge u_{j+1})) \\
&= g\left(\sum_{i=1}^{j+1} (-1)^{i+1} u_i f(u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_{j+1})\right. \\
&\qquad \left. + \sum_{r<t} (-1)^{r+t} f([u_r, u_t] \wedge u_1 \wedge \cdots \wedge \hat{u}_r \wedge \cdots \wedge \hat{u}_t \wedge \cdots \wedge u_{j+1})\right)
\end{aligned}$$

$$\begin{aligned}
&\partial^j(\Phi_j(f))(u_1 \wedge \cdots \wedge u_{j+1} \otimes g) \\
&= \Phi_j(f)\left(\sum_{i=1}^{j+1} (-1)^i u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_{j+1} \otimes u_i g\right. \\
&\qquad \left. + \sum_{r<t} (-1)^{r+t} [u_r, u_t] \wedge \cdots \wedge \hat{u}_r \wedge \cdots \wedge \hat{u}_t \wedge \cdots \wedge u_{j+1} \otimes g\right) \\
&= \sum_{i=1}^{j+1} (-1)^i (u_i g)(f(u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_{j+1})) \\
&\qquad + g\left(f\left(\sum_{r<t} (-1)^{r+t} [u_r, u_t] \wedge u_1 \wedge \cdots\right.\right. \\
&\qquad \left.\left. \wedge \hat{u}_r \wedge \cdots \wedge \hat{u}_t \wedge \cdots \wedge u_{j+1}\right)\right) \\
&= g\left(\sum_{i=1}^{j+1} (-1)^{i+1} u_i f(u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_{j+1})\right. \\
&\qquad \left. + \sum_{r<t} (-1)^{r+t} f([u_r, u_t] \wedge u_1 \wedge \cdots \wedge \hat{u}_r \wedge \cdots \wedge \hat{u}_t \wedge \cdots \wedge u_{j+1})\right)
\end{aligned}$$

Hence $\Phi_{j+1} \circ d^j = \partial^j \circ \Phi_j$. Q.E.D.

Let $\omega: \mathbf{g} \to \mathbf{g}$ be the Cartan involution (see [1, Chap. 1]) defined by $\omega(h) = -h$, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $i = 1, ..., n$. Then $\omega(\mathbf{r}) = \mathbf{r}$, $\omega(\mathbf{u}^{\pm}) = \mathbf{u}^{\mp}$. It induces an automorphism on the exterior algebra $\Lambda\mathbf{g}$ of $\mathbf{g}$, which we denote by $\wedge\omega$. Then $\wedge\omega$ maps $\Lambda^j\mathbf{u}^{\pm}$ to $\Lambda^j\mathbf{u}^{\mp}$.

Now we compose ω with the original representation of $\mathbf{g}$ (or $\mathbf{r}$) on $M(\lambda)^*$ (or $\Lambda^j\mathbf{u}^+ \otimes M(\lambda)^*$). We obtain a new representation of $\mathbf{g}$(or $\mathbf{r}$). Since $M(\lambda)^*$ is an irreducible lowest weight $\mathbf{g}$-module with the lowest weight $-\lambda$, it becomes an irreducible highest weight module with the highest weight λ, under the new representation of $\mathbf{g}$. We have a $\mathbf{g}$-isomorphism,

$$\eta: M(\lambda) \to M(\lambda)^*.$$

LEMMA 11. *With the new representation of* $\mathbf{r}$ *on* $\Lambda^j\mathbf{u}^+ \otimes M(\lambda)^*$, *we have* $\mathbf{r}$-*isomorphisms,*

$$\Psi_j = \wedge\omega \otimes \eta: \Lambda^j\mathbf{u}^- \otimes M(\lambda) \to \Lambda^j\mathbf{u}^+ \otimes M(\lambda)^*,$$

and

$$\begin{tikzcd} \Lambda^j\mathbf{u}^- \otimes M(\lambda) \arrow[r, "\Psi_j"] \arrow[d, "d_j"'] & \Lambda^j\mathbf{u}^+ \otimes M(\lambda)^* \arrow[d, "\partial_j"] \\ \Lambda^{j-1}\mathbf{u}^- \otimes M(\lambda) \arrow[r, "\Psi_{j-1}"'] & \Lambda^{j-1}\mathbf{u}^+ \otimes M(\lambda)^* \end{tikzcd}$$

is commutative.

Proof. All assertions follow directly from the fact that ω is an automorphism of $\mathbf{g}$, and $\wedge\omega$ an automorphism of the exterior algebra $\Lambda\mathbf{g}$.
Q.E.D.

We use $H_*(\mathbf{u}^-, M(\lambda))$, $H^*(\mathbf{u}^+, M(\lambda))$, $H_*(\mathbf{u}^+, M(\lambda)^*)$, $H^*(\mathbf{u}^+, M(\lambda))^*)$ for the homology of complex of cocomplex (1), (2), (3), (4). Since $\Lambda^j\mathbf{u}^- \otimes M(\lambda)$ is in the category $C(\mathbf{r})$ (see Section 1), we can assume $H_j(\mathbf{u}^-, M(\lambda)) = \sum_\mu m(\mu)$. By Lemma 11, $H_j(\mathbf{u}^+, M(\lambda)^*) = \sum_\mu w(-\mu)$, under the original representation, where $w(-\mu)$ are the irreducible lowest weight $\mathbf{r}$-modules with the lowest weights $-\mu$. By Lemmas 9 and 10,

$$H^j(\mathbf{u}^+, M(\lambda)^*) \cong (H_j(\mathbf{u}^+, M(\lambda)^*))^* = \sum_\mu (w(-\mu))^* = \sum_\mu m(\mu)$$

$$H^j(\mathbf{u}^+, M(\lambda)) \cong H^j(\mathbf{u}^+, M(\lambda)^*).$$

So we obtain

PROPOSITION 12. $H^j(\mathbf{u}^+, M(\lambda)) \cong H_j(\mathbf{u}^-, M(\lambda))$, *for any* j.

From now on, we will only discuss the complex (1), and its homology $H_*(\mathbf{u}^-, M(\lambda))$.

5. Kostant's Theorem and Bott's Theorem

In this section we will prove Kostant's formula. We will use the method in the appendix of [4; 3, Sect. 4] applying Casimir operators of **g** and **r** on complexes (6) and (7) (to be defined later), respectively. These computations will help us obtain the final results.

Let $\mathscr{G}$, $\mathscr{P}$, $\mathscr{U}^-$ be the universal enveloping algebras of the Lie algebra **g**, $\mathbf{p}=\mathbf{r}+\mathbf{u}^+$ and $\mathbf{u}^-$. Then $\mathscr{U}^- \subseteq \mathscr{G}$, $\mathscr{P} \subseteq \mathscr{G}$. The spaces $\Lambda^j(\mathbf{g}/\mathbf{p})$ are **p**-modules isomorphic to $\Lambda^j\mathbf{u}^-$ as **r**-modules, for $j=0, 1, \ldots$. Also $\mathscr{G} \otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p})$ are **g**-modules, where $x(a\otimes u)$ is defined by $xa\otimes u$ for $x\in\mathbf{g}$, $a\in\mathscr{G}$, $u\in\Lambda^j(\mathbf{g}/\mathbf{p})$. We have a **g**-module complex,

$$\cdots \longrightarrow \mathscr{G}\otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p}) \xrightarrow{d_j} \mathscr{G}\otimes_{\mathscr{P}} \Lambda^{j-1}(\mathbf{g}/\mathbf{p}) \longrightarrow \cdots$$

$$\longrightarrow \mathscr{G}\otimes_{\mathscr{P}} \Lambda^0(\mathbf{g}/\mathbf{p}) \xrightarrow{\varepsilon_0} C \longrightarrow 0, \tag{5}$$

where

$$\bar{d}_j(x\otimes \bar{y}_1 \wedge \cdots \wedge \bar{y}_j) = \sum_{i=1}^{j} (-1)^{i+1} xy_i \otimes \bar{y}_1 \wedge \cdots \wedge \hat{\bar{y}}_i \wedge \cdots \wedge \bar{y}_j$$

$$+ \sum_{r<t} (-1)^{r+t} x\otimes \overline{[y_r, y_t]}$$

$$\wedge\, \bar{y}_1 \wedge \cdots \wedge \hat{\bar{y}}_r \wedge \cdots \wedge \hat{\bar{y}}_t \wedge \cdots \wedge \bar{y}_j$$

for $x\in\mathscr{G}$, $y_1, \ldots, y_j \in \mathbf{g}$, $\bar{y}_i = y_i + \mathbf{p} \in \mathbf{g}/\mathbf{p}$ and ε_0 maps $x\in\mathscr{G}$ to its constant term.

It is very easy to check that $\bar{d}_j$, $j=1, 2, \ldots, \varepsilon_0$, are well defined **g**-module homomorphisms, and $\bar{d}_j \circ \bar{d}_{j+1} = 0$, $\varepsilon_0 \circ \bar{d}_1 = 0$ (see [4]).

Proposition 13. *Complex* (5) *is exact.*

Proof. See [4, Sect. 1]. Q.E.D.

Let $Y_j = (\mathscr{G}\otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p}))\otimes M(\lambda)$. We obtain an exact **g**-module complex,

$$\cdots \longrightarrow Y_j \xrightarrow{D_j} Y_{j-1} \longrightarrow \cdots \longrightarrow Y_0 \xrightarrow{\varepsilon} M(\lambda) \longrightarrow 0, \tag{6}$$

where $D_j = \bar{d}_j \otimes 1$, $j=1, 2, \ldots, \varepsilon = \varepsilon_0 \otimes 1$.

Let the complex number field C be a trivial right $\mathbf{u}^-$-module. Consider $X_j = C\otimes_{\mathscr{U}^-} Y_j = C\otimes_{\mathscr{U}^-} ((\mathscr{G}\otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p}))\otimes M(\lambda))$. Since $\mathbf{u}^-$ is **r**-invariant and $\mathbf{u}^-$ kills C, we can give X_j an $\mathbf{r}$-module structure. $x(a\otimes y)=a\otimes xy$, $x\in \mathbf{r}$, $a\in C$, $y\in Y_j$. We obtain an $\mathbf{r}$-module complex,

$$\cdots\longrightarrow X_j \xrightarrow{D'_j} X_{j-1}\longrightarrow\cdots\xrightarrow{D'_1} X_0\longrightarrow 0, \tag{7}$$

where $D'_j=1\otimes D_j$, $j=1, 2, \ldots$.

By the P–B–W theorem, we have linear isomorphisms,

$$\Theta_j\colon \Lambda^j\mathbf{u}^-\otimes M(\lambda)\to X_j;\ u_1\wedge\cdots\wedge u_j\otimes v\mapsto 1\otimes((1\otimes\bar{u}_1\wedge\cdots\wedge\bar{u}_j)\otimes v),$$

where $u_1, \ldots, u_j\in\mathbf{u}^-$, $v\in M(\lambda)$, and it is easy to check that Θ_j, $j=1, 2, \ldots$ are $\mathbf{r}$-isomorphisms.

LEMMA 14.

$$\begin{tikzcd}
\Lambda^j\mathbf{u}^-\otimes M(\lambda) \arrow[r, "\Theta_j"] \arrow[d, "d_j"'] & X_j \arrow[d, "D'_j"] \\
\Lambda^{j-1}\mathbf{u}^-\otimes M(\lambda) \arrow[r, "\Theta_{j-1}"] & X_{j-1}
\end{tikzcd}$$

is commutative.

Proof. This can be checked directly. Q.E.D.

Then the homology of complex (1) is isomorphic to the homology of complex (7) as $\mathbf{r}$-modules. Next we show that every Y_j is a free $\mathscr{U}^-$-module.

LEMMA 15. *Let L be a Lie algebra, $\mathscr{U}(L)$ its universal enveloping algebra. Let M, N be two L-modules. If M is a free $\mathscr{U}(L)$-module, so is $M\otimes N$.*

One need only prove:

LEMMA 16. *Let L, $\mathscr{U}(L)$, N be same as in Lemma 14. We can give $\mathscr{U}(L)\otimes N$ two L-module structures. For $x\in L$, $u\in\mathscr{U}(L)$, $n\in N$,*

(1) $x\circ(u\otimes n)=^{\text{def}} xu\otimes n.$

(2) $x*(u\otimes n)=^{\text{def}} xu\otimes n+u\otimes xn.$

Then $\{\mathscr{U}(L)\otimes N, \circ\}\cong\{\mathscr{U}(L)\otimes N, *\}$.

(*There is a more generalized result in* [4, *Sect.* 1]; *this lemma is a special case of that result.*)

Proof. The conclusion is a general one for Hopf algebras.

If s denotes the antipode (in $\mathscr{U}(L)$, $s(x)=-x$, for all $x\in L$) and

$\Delta: \mathscr{U}(L) \to \mathscr{U}(L) \otimes \mathscr{U}(L)$ the coproduct $(\Delta(x) = x \otimes 1 + 1 \otimes x$, for all $x \in L)$, then the map $f: \{\mathscr{U}(L) \otimes N, \circ\} \to \{\mathscr{U}(L) \otimes N, *\}$ with $f(u \otimes n) = \sum u^{(1)} \otimes u^{(2)} n$, where $\Delta(u) = \sum u^{(1)} \otimes u^{(2)}$, u, $u^{(i)} \in \mathscr{U}(L)$, $n \in N$, is a module map.

Likewise, the map $g: \{\mathscr{U}(L) \otimes N, *\} \to \{\mathscr{U}(L) \otimes N, \circ\}$ with $g(u \otimes n) = \sum u^{(1)} \otimes s(u^{(2)}) n$ is a module map. And these maps are inverses to one another. Q.E.D.

Now Lemma 15 follows by the fact that $\{\mathscr{U}(L) \otimes N, \circ\}$ is obviously a free $\mathscr{U}(L)$-module.

Since $\mathscr{G} \otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p}) \cong \mathscr{U}^- \otimes \Lambda^j(\mathbf{u}^-)$, as $\mathbf{u}^-$-module, by Lemma 15, $Y_j = (\mathscr{G} \otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p})) \otimes M(\lambda)$ is a free $\mathscr{U}^-$-module. By definition of X_j, $j = 0, 1, \ldots,$ we have surjective linear maps,

$$q_j: Y_j \to X_j, \qquad y \mapsto 1 \otimes y, \qquad \text{for} \quad y \in Y_j.$$

And it is easy to see that

$$\begin{array}{ccc} Y_j & \xrightarrow{D_j} & Y_{j-1} \\ {\scriptstyle q_j}\big\downarrow & & \big\downarrow{\scriptstyle q_{j-1}} \\ X_j & \xrightarrow[D_j']{} & X_{j-1} \end{array}$$

is commutative. Let $\Omega^+ (\Omega_S^+)$ be the set of all roots in $\Delta^+ (\Delta_S^+)$, with multiplicities. We define the Casimir operators Γ and γ of the Lie algebras $\mathbf{g}$, $\mathbf{r}$, respectively.

$$\Gamma = 2 \sum_{\alpha \in \Omega^+} e_{-\alpha} e_a + \sum_{i=1}^{\dim \mathbf{h}} k_i h_i + 2\nu^{-1}(\rho)$$

$$\gamma = 2 \sum_{\alpha \in \Omega_S^+} e_{-\alpha} e_\alpha + \sum_{i=1}^{\dim \mathbf{h}} k_i h_i + 2\nu^{-1}(\rho),$$

where $e_{\pm\alpha} \in \mathbf{g}^{\pm\alpha}$, k_i, $h_i \in \mathbf{h}$, their choice depending on the bilinear form $(\cdot, \cdot)$ of $\mathbf{g}$ (see [1, Chap. 2] for details). Since Y_j, X_j, $M(\lambda)$ are in the category $\mathscr{O}$, we apply Γ, γ on complexes (6), (7), respectively. They commute with the boundary operators.

LEMMA 17.

$$\begin{array}{ccc} Y_j & \xrightarrow{\Gamma} & Y_j \\ {\scriptstyle q_j}\big\downarrow & & \big\downarrow{\scriptstyle q_j} \\ X_j & \xrightarrow[\gamma]{} & X_j \end{array}$$

is commutative.

Proof. For any $\alpha \in \Delta^+(S)$, $e_{-\alpha}$ is contained in $\mathbf{u}^-$. Then $1 \otimes e_{-\alpha} e_\alpha y = 1 e_{-\alpha} \otimes e_\alpha y = 0$, for any $y \in Y_j$. Hence $q_j \circ \Gamma = \gamma \circ q_j$. Q.E.D.

For $c \in C$, let

$$Y_j^c = \{ y \in Y_j \mid (\Gamma - c)^n y = 0, \text{ for some nonnegative integer } n \}$$
$$X_j^c = \{ x \in X_j \mid (\gamma - c)^n x = 0, \text{ for some nonnegative integer } n \}.$$

Since every weight space of Y_j is finite dimensional, Γ stabilizes each weight space. Then $Y_j = \bigoplus_{c \in C} Y_j^c$. Similarly $X_j = \bigoplus_{c \in C} X_j^c$. Let $c_0 = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$, $Y_j^\times = \sum_{c \neq c_0} Y_j^c$, $X_j^\times = \sum_{c \neq c_0} X_j^c$. Since Γ acts on $M(\lambda)$ by the constant c_0, the complex (6) is the direct sum of the following two complexes:

$$\cdots \to Y_j^{c_0} \to \cdots \to Y_0^{c_0} \to M(\lambda) \to 0 \tag{8}$$

$$\cdots \to Y_j^\times \to \cdots \to Y_0^\times \to 0. \tag{9}$$

Complex (7) is the direct sum of the following two complexes:

$$\cdots \to X_j^{c_0} \to \cdots \to X_0^{c_0} \to 0 \tag{10}$$

$$\cdots \to X_j^\times \to \cdots \to X_0^\times \to 0. \tag{11}$$

By Lemma 17, $X_j^{c_0} = q_j(Y_j^{c_0}) = C \otimes_{\mathscr{U}^-} Y_j^{c_0}$ and $X_j^\times = q_j(Y_j^\times) = C \otimes_{\mathscr{U}^-} Y_j^\times$. Every $Y_j^\times$ is a projective $\mathscr{U}^-$-module, so complex (9) is a projective resolution of 0. This implies that the homology of complex (11) is 0 (see [6, Chap. 6]. Hence the homology of complex (7) is equal to the homology of complex (10).

PROPOSITION 18. *Every irreducible component of $H_j(\mathbf{u}^-, M(\lambda))$ is of the form $m(\mu)$, $\mu \in P_S$, with $(\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$.*

Proof. $\Lambda^j \mathbf{u}^- \otimes M(\lambda)$ is in category $C(\mathbf{r})$. If $m(\mu)$ is a component of $H_j(\mathbf{u}^-, M(\lambda))$, it must be a component of $X_j^{c_0}$. γ is the constant $(\mu + \rho, \mu + \rho) - (\rho, \rho)$ on $m(\mu)$, and is the constant c_0 on $X_j^{c_0}$. Hence $(\mu + \rho, \mu + \rho) - (\rho, \rho) = c_0$, $(\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$. Q.E.D.

LEMMA 19. *μ is weight of $\Lambda \mathbf{n}^-$, if and only if $\mu + \rho$ is a weight of $M(\rho)$. In this case μ and $\mu + \rho$ have the same multiplicity.*

Proof. By Weyl–Kac's character formula and denominator identity,

$$ch_{\mathfrak{h}} M(\rho) = \frac{\sum_{w \in W} \det(w)\, e(w(2\rho))}{\sum_{w \in W} \det(w)\, e(w(\rho))}.$$

$$\sum_{w \in W} \det(w)\, e(w(\rho)) = e(\rho) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{\dim \mathfrak{g}^\alpha}.$$

$$\Rightarrow \sum_{w \in W} \det(w)\, e(w(2\rho)) = e(2\rho) \prod_{\alpha \in \Delta^+} (1 - e(-2\alpha))^{\dim \mathfrak{g}^\alpha},$$

since $\beta_1 + \cdots + \beta_s = \gamma$ iff $2\beta_1 + \cdots + 2\beta_s = 2\gamma$, for $\beta_1, ..., \beta_s \in \Delta^+$. And $\prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{\dim \mathfrak{g}^\alpha}$ is invertible, so is $\sum_{w \in W} \det(w)\, e(w(\rho))$, $\Rightarrow ch_{\mathfrak{h}} M(\rho) = e(\rho) \prod_{\alpha \in \Delta^+} (1 + e(-\alpha))^{\dim \mathfrak{g}^\alpha} = e(\rho)\, ch_{\mathfrak{h}} \Lambda \mathbf{n}^-$. Q.E.D.

By the way, we have $ch_{\mathfrak{h}}(M(\rho) \otimes M(\lambda)) = e(\rho)\, ch_{\mathfrak{h}}(\Lambda \mathbf{n}^- \otimes M(\lambda))$. μ is a weight of $\Lambda \mathbf{n}^- \otimes M(\lambda)$ if and only if $\mu + \rho$ is a weight of $M(\rho) \otimes M(\lambda)$, and in this case they have the same multiplicity.

Lemma 20. *If μ is a weight of $\Lambda^j \mathbf{u}^- \otimes M(\lambda)$, and $(\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$, then*

(1) *There exists a $w \in W(S)$ with $l(w) = j$, such that $w(\lambda + \rho) - \rho = \mu$.*

(2) *The multiplicity of μ in $\Lambda^j \mathbf{u}^- \otimes M(\lambda)$ is 1.*

Proof. By Lemma 19, $\mu + \rho$ is a weight of $M(\rho) \otimes M(\lambda)$ with the same multiplicity as μ in $\Lambda \mathbf{n}^- \otimes M(\lambda)$. Choose a $\bar{w} \in W$ such that $\bar{w}(\mu + \rho)$ is a dominant weight. We can assume that $\bar{w}(\mu + \rho) = \rho + \lambda - \varphi$, where φ is a sum of some positive roots. Then

$$\begin{aligned}(\lambda + \rho, \lambda + \rho) &= (\mu + \rho, \mu + \rho) = (\bar{w}(\mu + \rho), \bar{w}(\mu + \rho)) \\ &= (\lambda + \rho - \varphi, \lambda + \rho - \varphi) \\ &= (\lambda + \rho, \lambda + \rho) - (\lambda + \rho, \varphi) - (\lambda + \rho - \varphi, \varphi).\end{aligned}$$

Since $\lambda + \rho - \varphi$ is dominant, and $\lambda + \rho$ is strongly dominant, $\varphi = 0$. Let $w = \bar{w}^{-1}$. Then $\mu = w(\lambda + \rho) - \rho$. It is clear that the multiplicity of $\mu + \rho = w(\lambda + \rho)$ in $M(\rho) \otimes M(\lambda)$ is 1. Then the multiplicity of μ in $\Lambda \mathbf{n}^- \otimes M(\lambda)$ is 1. If $\Phi_w = \beta_1, ..., \beta_t$, choose $0 \neq u_i \in \mathfrak{g}^{-\beta_i}$, $0 \neq x \in M(\lambda)_{w(\lambda)}$. Then $u_1 \wedge \cdots \wedge u_t \otimes x$ is a vector of weight μ (see Lemma 4). Since μ is a weight of $\Lambda^j \mathbf{u}^- \otimes M(\lambda)$ and the multiplicity of μ in $\Lambda \mathbf{n}^- \otimes M(\lambda)$ is 1, $u_1 \wedge \cdots \wedge u_t \otimes x \in \Lambda^j \mathbf{u}^- \otimes M(\lambda)$. This implies $l(w) = t = j$, $\beta_i \in \Delta^+(S)$, $i = 1, ..., t$, $w \in W(S)$. Q.E.D.

COROLLARY 21. *If $m(\mu)$ is an irreducible component of $H_j(\mathbf{u}^-, M(\lambda))$, then*

(1) *$\mu = w(\lambda+\rho)-\rho$, for some $w \in W(S)$ with $l(w)=j$.*

(2) *$H_j(\mathbf{u}^-, M(\lambda))$ has only one copy of $m(\mu)$ in itself.*

Proof. By Proposition 18 and Lemma 20. Q.E.D.

THEOREM 22 (Kostant's Formula). $H^j(\mathbf{u}^+, M(\lambda)) = H_j(\mathbf{u}^-, M(\lambda)) = \sum_{w\in W(S),\, l(w)=j} m(w(\lambda+\rho-\rho)$.

Proof. By Corollary 21, $H_j(\mathbf{u}^-, M(\lambda))$ is at most equal to $\sum_{w\in W(S),\, l(w)=j} m(w(\lambda+\rho)-\rho)$. By Corollary 8 we have

$$\sum_{j=1}^{\infty} (-1)^j \, ch_{\mathbf{r}} H_j(\mathbf{u}^-, M(\lambda)) = \sum_{w\in W(S)} \det(w)\, ch_{\mathbf{r}} m(w(\lambda+\rho)-\rho).$$

For different $w \in W(S)$, we obtain different $w(\lambda+\rho)-\rho$ (see Lemma 4), hence

$$H_j(\mathbf{u}^-, M(\lambda)) = \sum_{\substack{w\in W(S)\\ l(w)=j}} m(w(\lambda+\rho)-\rho). \qquad \text{Q.E.D.}$$

Note. In fact, we proved $X_j^{co} = \sum_{w\in W(S),\, l(w)=j} m(w(\lambda+\rho)-\rho)$.

This implies the boundary map from X_j^{co} to X_{j-1}^{co} is the zero map, and makes resolution (8) of $M(\lambda)$ interesting.

COROLLARY 23 (Bott's Formula). $\dim H^j(\mathbf{n}^+, M(\lambda)) = \dim H_j(\mathbf{n}^-, M(\lambda))$ $=$ *The number of elements $w \in W$ with $l(w)=j$.*

Proof. Apply Kostant's formula in the case of $S=\varnothing$. Q.E.D.

6. RESOLUTION OF $M(\lambda)$

Reference [4, Theorem 8.7] proves that in the case where $\mathbf{r}$ has finite dimension every term in the resolution

$$\cdots \to Y_j^{co} \to \cdots \to Y_0^{co} \to M(\lambda) \to 0 \tag{8}$$

has a very nice structure. As $\mathbf{g}$-module, $Y_j^{co} = \bigcup_{i=1}^{n_j} V_i$, $V_i/V_{i-1} \cong V^{m(\mu_i)}$ the generalized Verma $\mathbf{g}$-module induced from irreducible $\mathbf{r}$-module $m(\mu_i)$, where,

$$n_j = {}^{\#}w \in W(S) \text{ with } l(w) = j$$

$$\{\mu_1, ..., u_{n_j}\} = \{w(\lambda+\rho) - \rho \mid w \in W(S), l(w) = j\}.$$

As $\mathbf{r}$-modules, $Y_j^{co} \cong \bigoplus_{i=1}^{n_j} V^{m(\mu_i)}$.

Using Theorem 22 of the last section we can prove the corresponding result in our case. First, we have to borrow Proposition 1.7 from [4], which implies

$$Y_j = (\mathscr{G} \otimes_{\mathscr{P}} \Lambda^j(\mathbf{g}/\mathbf{p})) \otimes M(\lambda) \cong \mathscr{G} \otimes_{\mathscr{P}} (\Lambda^j(\mathbf{g}/\mathbf{p} \otimes M(\lambda)).$$

In the $\mathbf{p}$-module $\Lambda^j(\mathbf{g}/\mathbf{p}) \otimes M(\lambda)$, let N_k be the sum of the weight spaces $(\Lambda^j(\mathbf{g}/\mathbf{p}) \otimes M(\lambda))_\beta$, with $\beta = \lambda - \sum_{i=1}^n k_i \alpha_i$, and $\sum_{i=s+1}^n k_i \leqslant k$. Then $\Lambda^j(\mathbf{g}/\mathbf{p}) \otimes M(\lambda) = \bigcup_{k=0}^{\infty} N_k$, and $0 \subseteq N_0 \subseteq \cdots \subseteq N_k \subseteq N_{k+1} \subseteq \cdots$ gives a filtration on $\Lambda^j(\mathbf{g}/\mathbf{p}) \otimes M(\lambda)$. It is clear that all N_k are $\mathbf{p}$-modules, and $\mathbf{u}^+ N_k \subseteq N_{k-1}$. As $\mathbf{r}$-modules, N_k/N_{k-1} belongs to the category $C(\mathbf{r})$ (see Section 2), therefore it is a direct sum of $m(\mu)$, $\mu \in P_S$. (The number of such $m(\mu)$ may not be finite, but any way is is countable.). This gives a refinement to the filtration $\{N_k\}$. So, we could assume that there is an index set $\{(m, k)\}$, where k ranges over $\{0, 1, 2, ...\}$, m ranges over $\{0, 1, ..., t\}$, where t could be a non-negative integer or $+\infty$, such that $\Lambda^j(\mathbf{g}/\mathbf{p}) \otimes M(\lambda) = \bigcup_{(m,k)} N_{m,k}$, satisfies:

(1) $N_{m,k} \subseteq N_{m,k+1}$, $N_{m,k} \subseteq N_{m+1,k'}$, for any m, k, k'.

(2) $N_{m+1,0} = \bigcup_{k=0}^{+\infty} N_{m,k}$, for any m.

(3) $\mathbf{u}^+ N_{m,k+1} \subseteq N_{m,k}$, for any m, k; $\mathbf{u}^+ N_{m+1,0} \subseteq N_{m,k'}$, for some k'.

(4) As $\mathbf{r}$-module $N_{m,k}/N_{m,k-1} \cong m(\mu_{m,k})$, for some $\mu_{m,k} \in P_S$, $k \geqslant 1$.

Now, let $V_{m,k} = \mathscr{G} \otimes_{\mathscr{P}} N_{m,k}$. This gives a filtration on $Y_j = \mathscr{G} \otimes_{\mathscr{P}} (\Lambda^j(\mathbf{g}/\mathbf{p}) \otimes M(\lambda))$, $Y_j = \bigcup_{(m,k)} V_{m,k}$, satisfying:

(1′) $V_{m,k} \subseteq V_{m,k+1}$, $V_{m,k} \subseteq V_{m+1,k'}$, for any m, k, k'.

(2′) $V_{m+1,0} = \bigcup_{k=0}^{+\infty} V_{m,k}$, for any m.

(3′) As $\mathbf{g}$-module $V_{m,k}/V_{m,k-1} \cong V^{m(\mu_{m,k})}$, $k \geqslant 1$.

These filtrations are not as good as those in [4]. Since the subset S of $\{1, ..., n\}$ can be arbitrary, our approach makes the index set $\{(m, k)\}$ a little more complicated. For this reason we are unable to describe the structure of Y_j^{co} before Theorem 22.

The Casimir operator Γ splits Y_j and each $V_{m,k}$ into two parts, $Y_j = Y_j^{co} \oplus Y_j^{\times}$, $V_{m,k} = V_{m,k}^{co} \oplus V_{m,k}^{\times}$. So Y_j^{co} yields a filtration from Y_j, $Y_j^{co} = \bigcup_{(m,k)} V_{m,k}^{co}$, which satisfies:

(I) $V^{c0}_{m,k}/V^{c0}_{m,k-1} \cong V_{m,k}/V_{m,k-1} \cong V^{m(\mu_{m,k})}$, if $|\mu_{m,k}+\rho|^2 = |\lambda+\rho|^2$.

(II) $V^{c0}_{m,k} = V^{c0}_{m,k-1}$, if $|\mu_{m,k}+\rho|^2 \neq |\lambda+\rho|^2$.

(III) $V^{c0}_{m+1,0} = V^{c0}_{m,k}$, if there exists a k such that for all $k'>k$, $|\mu_{m,k'}+\rho|^2 \neq |\lambda+\rho|^2$.

After taking out extra terms in the filtration of Y^{c0}_j, we obtain a filtration of Y^{c0}_j in the usual sense.

LEMMA 24. *The filtration of Y^{c0}_j is finite.*

Proof. If it were infinite, we could choose infinite terms

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k \subseteq \cdots$$

from the filtration, such that $V_k/V_{k-1} \cong V^{m(\mu_k)}$, $\mu_k \in P_S$. Reference [4, Lemma 7.8] says $\bigcup_{k=0}^{+\infty} V_k \cong \bigoplus_{k=0}^{+\infty} V^{m(\mu_k)}$ as $\mathbf{r}+\mathbf{u}^-$-modules. Then $X^{c0}_j = C \otimes_{\mathscr{U}^-} Y^{c0}_j$ contains an $\mathbf{r}$-submodule $\bigoplus_{k=0}^{+\infty} m(\mu_k)$ which has infinitely many irreducible components, contradicting the Note of Theorem 22. Q.E.D.

Still using [4, Lemma 7.8] and the Note of Theorem 22, we have:

THEOREM 25. *Let $\{w_1, ..., w_{n_j}\} = \{w \in W(S) \mid l(w) = j\}$. The jth term Y^{c0}_j in the resolution* (8)

$$\cdots \to Y^{c0}_j \to \cdots \to Y^{c0}_0 \to M(\lambda) \to 0$$

has a filtration of $\mathbf{g}$*-submodules,*

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{n_j}.$$

For any k, N_k/N_{k-1} is isomorphic to the generalized Verma module $V^{m(w_k(\lambda+\rho)-\rho)}$ induced from $m(\omega_k(\lambda+\rho)-\rho)$. As the $\mathbf{r}+\mathbf{u}^-$-module, Y^{c0}_j is the direct sum of these $V^{m(\omega_k(\lambda+\rho)-\rho)}$.

This theorem is an analogue of [4, Theorem 8.7].

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